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Casimir effect in background of static domain wall

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In this paper we investigate the vacuum expectation values of energy- momentum tensor for conformally coupled scalar field in the standard parallel plate geometry with Dirichlet boundary conditions and on background of planar domain wall case. First we calculate the vacuum expectation values of energy-momentum tensor by using the mode sums, then we show that corresponding properties can be obtained by using the conformal properties of the problem. The vacuum expectation values of energy-momentum tensor contains two terms which come from the boundary conditions and the the gravitational background. In the Minkovskian limit our results agree with those obtaind in [3].

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1. Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory [1], [2]. Since it's first prediction by Casimir in 1948 [3] this effect is investigated for various cases of boundary geometries and various types of fields. The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of external gravitational field. In this paper we shall consider a simple example when these two types of sources for vacuum polarization are present. We investigate the vacuum expectation values of the energy-momentum tensor for conformally coupled scalar field in the standard parallel plate geometry with Dirichlet boundary conditions and on background of planar static domain wall case. It has been shown in [4] and [5], that the gravitational field of the vacuum domain wall with a source of the form

$$T_\mu^\nu = \sigma\delta(x) diag(1,\, 0,\, 1,\, 1) \tag{1}$$

does not correspond to any exact static solution of Einstein equations (on domain wall solutions of Einstein-scalar-field equations see [6]). However the static solutions can be constructed in presence of an additional background energy-momentum tensor. Such a type solution has been fond in [7]. First we calculate the vacuum expectation values of energy-momentum tensor by using the mode sums. We obtain the result as a direct sum of two terms: boundary term and term which presents the vacuum polarization in the domain wall geometry in the case of absence of boundaries. It is shown that boundary part of total energy between the plates and corresponding pressures on plates are related by standard thermodynamical relation. Then we show that corresponding properties can be obtained by using the conformal properties of the problem.

2. Vacuum expectation values of energy-momentum tensor

In this paper we shall consider the conformally coupled real scalar field ϕ, which satisfies

$$(\Box + \frac{1}{6}R)\phi = 0, \qquad \Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \tag{2}$$

and propagates on background of gravitational field generated by static domain wall solution from [7]. The corresponding metric has the form

$$ds^2 = A^{-2\alpha}(dt^2 - dy^2 - dz^2) - A^{-2(\alpha+\gamma+1)}dx^2, \qquad A = A(x) \equiv 1 + K|x|, \tag{3}$$

where $\alpha > 0, K > 0$. The equation (3) describes a planar domain wall with energy-momentum tensor $T_\mu^\nu = \delta(x) diag(1, 0, 1, 1)$ in the background field with Ricci tensor

$$R_\mu^\nu = \alpha(\gamma - 2\alpha)K^2 A^{2(\alpha+\gamma)} diag(1, \frac{3\gamma}{\gamma - 2\alpha}, 1, 1) \tag{4}$$

Note that for the energy density of the background to be positive we must have $\gamma < \alpha/2$. In what follows as a boundary configuration we shall consider two plates parallel to each other and to domain wall, with x coordinates equal to x_1 and x_2 (to be definit we shall consider right half space of domain wall geometry $x_1, x_2 > 0$). For the points on plate the scalar field obeys Dirichlet boundary condition

$$\phi(x = x_1) = \phi(x = x_2) = 0 \tag{5}$$

The quantization of field (2) on background of equation (3) is standard let $\phi_\alpha^{(\pm)}(x)$ be complete set of orthonormalized positive and negative frequency solutions to the field equation (2), obeying boundary conditions (5). The canonical quantization can be done by expanding the general solution of (2) in terms of $\phi_\alpha^{(\pm)}$,

$$\phi = \sum_\alpha (\phi_\alpha^+ a_\alpha + \phi_\alpha a_\alpha^{(+)}) \tag{6}$$

and declearing the coeficients a_α ,a_α^+ as operators satisfying standard commutation relation for bosonic fields. The vacuum state $|0>$ is defiend as $a_\alpha|0>=0$. This state is different from the vacuum state for domain wall geometry without bondaries, $|\bar{0}>$. To investigate effects due to the presence of boundaries we shall consider vacuum expectation values of energy-momentum tensor operator, $<0|T_{\mu\nu}|0>$. By substituting the expansion (6) and using the definition of vacuum state it can be easly seen that [8]

$$<0|T_{\mu\nu}|0> = \sum_\alpha T_{\mu\nu}\{\phi_\alpha^{(+)}, \phi_\alpha^{(-)}\} \tag{7}$$

Here on the rhs the bilinear form $T_{\mu\nu}\{\phi, \psi\}$ is determined by the classical energy-momentum tensor for conformally coupled scalar field (see for exampel [8]). To calculate the vacuum expectation values by (7) we need the explicit form of eigenfunctions $\phi_\alpha^{(\pm)}$. For this case the metric and boundary conditions are static and translation invariant in directions parallel to the domain wall. It follows from here that the corresponding part has standard plane wave structure:

$$\phi_\alpha^\pm = \varphi(x) \exp\left[\pm i(k_y y + k_z z - \omega t)\right] \tag{8}$$

The equation for $\varphi(x)$ is obtained from the field equation (2) and for domain wall metric (3) has the form

$$\varphi^{''}(x) + (\gamma + 1 - 2\alpha)K sgn(x) A^{-1} \varphi^{'} + [\alpha(\alpha - \gamma)K^2 A^{-2} + k_x^2 A^{-2(\gamma+1)}]\varphi = 0 \tag{9}$$

with ${k_x}^2 = \omega^2 - {k_t}^2$, ${k_t}^2 = {k_y}^2 + {k_z}^2$. We shall consider the region between plates. The solution to equation (9) in this region obeying boundary conditions (5) is

$$\varphi(x) = const A^\alpha \sin(k_x v), \qquad k_x = \frac{n\pi}{a}, \qquad n = 1, 2, ... \tag{10}$$

with the relation

$$v(x) = \frac{[(1+K|x_1|)^{-\gamma} - (1+K|x|)^{-\gamma}]}{K\gamma}, \qquad a = v(x_2) \tag{11}$$

By using this relations and normalizing the eigenfunctions by standard way one obtains

$$\phi_\beta^{(\pm)}(t,\vec{r}) = \frac{A^\alpha}{2\pi\sqrt{\omega a}} \sin(k_x v) e^{\pm i(k_y y + k_z z - \omega t)}, \quad \beta = (n, k_y, k_z), \quad \omega^2 = \left(\frac{n\pi}{a}\right)^2 + k_t^2 \tag{12}$$

Before to start specific calculation with formula (7) it is convenient to present energy-momentum tensor for conformally coupeld scalar field in the form

$$T_{ik} = \partial_i\phi\partial_k\phi - \frac{1}{6}(\nabla_i\nabla_k + \frac{1}{2}g_{ik}\Box + R_{ik})\phi^2 \tag{13}$$

By using the equation (7) with this form of energy-momentum tensor and with eigenmodes (12) we receive

$$< 0|T_{ik}|0 > = < 0|\partial_i\phi\partial_k\phi|0 > -\frac{1}{6}(\nabla_i\nabla_k + \frac{1}{2}g_{ik}\Box + R_{ik}) < 0|\phi^2|0 > . \tag{14}$$

It is convenient first to calculate the quantity

$$< 0|\phi(x)\phi(x')|0 > =$$
$$\frac{A^\alpha(x)A^\alpha(x')}{4\pi^2 a} \int d^2k_t \sum_{n=1}^{\infty} \frac{1}{\omega} \sin(k_x v(x)) \sin(k_x v(x')) e^{i[k_y(y-y') + k_z(z-z') - \omega(t-t')]} \tag{15}$$

Using Abel-Plana summation formula

$$\sum_{n=1}^{\infty} f(n) = \int_0^\infty f(x)dx - \frac{1}{2}f(0) + i\int_0^\infty \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}dx \tag{16}$$

to sum over n, and after calculating arising integrals one obtains

$$< 0|\phi(x)\phi(x')|0 > = \frac{A^\alpha(x)A^\alpha(x')}{8a^2}\frac{\sinh(u)}{u}$$
$$\{[\cosh(u) - \cos(\frac{\pi}{a}(v(x) - v(x')))]^{-1} - [\cosh(u) - \cos(\frac{\pi}{a}(v(x) + v(x')))]^{-1}\} \tag{17}$$

where $u = \frac{\pi}{a}[(y-y')^2 + (z-z')^2 - (t-t')^2]^{1/2}$. The vacuum expectation value of energy-momentum tensor may be found now by using the relation

$$< 0|T_{ik}(x)|0 > = lim_{x\to x'} \hat{T}_{ik} < 0|\phi(x)\phi(x')|0 > \tag{18}$$

where the form of the second order operator $\hat{T}_{ik}$ is obvious from (14):

$$\hat{T}_{ik} = \partial_i \partial_k^{'} - \frac{1}{6}(\nabla_i \nabla_k + \frac{1}{2} g_{ik} \Box + R_{ik}) \tag{19}$$

and $\partial_k^{'} = \partial / \partial x_k^{'}$. The vacuum expectation value of course is infinite. This divergencies come from the first term in figure brackets of (17). In this paper we are mainly interested in quantum effects due to the existence of boundaries. Let $|\bar{0}>$ be the vacuum state for conformally coupled scalar field in the case of absence of boundaries. Let us consider the difference

$$< T_{ik}^{(b)}(x) > = < 0|T_{ik}(x)|0> - < \bar{0}|T_{ik}(x)|\bar{0}> =$$
$$lim_{x \to x'} \hat{T}_{ik} [< 0|\phi(x)\phi(x')|0> - < \bar{0}|\phi(x)\phi(x')|\bar{0}>] \tag{20}$$

This quantity describes the boundary contribution to the polarization of the vacuum and is finit. To see this note that the expression for $< \bar{0}|\phi(x)\phi(x')|\bar{0}>$ may be obtained from (17) by taking $a \to \infty$ and has the following form

$$< \bar{0}|\phi(x)\phi(x')|\bar{0}> = -\frac{A^{\alpha}(x) A^{\alpha}(x')}{4\pi^2} \left\{ [v(x) - v(x')]^2 + (y - y')^2 + (z - z')^2 - (t - t')^2 \right\} \tag{21}$$

As it can be easily seen the divergences in (17) and (21) cancel in calculating boundary contribution (20). Substitving (17) and (21) into (20) after some calculations and arising the second index one obtains for the region between plates

$$< T_i^{(b)k} > = -\frac{\pi^2 A^{4\alpha}}{1440 a^4} diag(1, -3, 1, 1), \qquad x_1 \leq x \leq x_2 \tag{22}$$

and zero outside of this region. Here a is defined by equation (11). Note that a is different from the proper distance a_p between the plates,

$$a_p = \int_{x_1}^{x_2} g_{11} dx_1 = \left[(1 + K x_1)^{-\alpha-\gamma} - (1 + K x_2)^{-\alpha-\gamma} \right] / k(\alpha + \gamma) \tag{23}$$

In calculating (22) all terms which come from derivatives $\partial A / \partial x$ and $\partial^2 A / \partial x^2$ and are proportional to K and K^2 are cancelled. As we shall see this is direct consequence of the conformal properties of metric and field under consideration. In the limit of no geravitation, $K \to 0$ from (22) we obtain standard Casimir result for parallel plate configuration. In the case of scalar field and $a = x_2 - x_1$. By using this result the vacuum expectation value for total energy-momentum tensor can be written as

$$< 0|T_{ik}|0> = < T_{ik}^{(b)} > + < T_{ik}^{(g)} > \tag{24}$$

where the second summand of right-hand side is the part describing the polarization of scalar vacuum by domain wall geravitational field in the case of absence of boundaries. All divergences are contained in this part. The corresponding regularization can be done by using the standard methods of quantum field theory in curved space-time (see, for example [8]). Most simply this can be done by using the conformally flatness of the metric (refmetric) (see [7] and below). In this cas the anomalous trace determines the total energy-momentum tensor (see [8]) and the regular part of purely gravity contribution to (24) is equal to

$$< T_{ik}^{(g)} >= reg < \bar{0}|T_{ik}^{(g)}|\bar{0} >= -\frac{\alpha + 4\gamma}{2880\pi^2}\alpha(3\gamma^2 - 2\alpha^2)A^{4(\alpha+\gamma)}diag(1, -\frac{3\alpha}{\alpha + 4\gamma}, 1, 1) \quad (25)$$

now from (24) and (25) it follows that regularized total energy-momentum tensor in the region between plates are given by

$$< T_i^k >=< T_i^{(b)k} > + < T_i^{(g)k} > \quad (26)$$

where boundary, $< T_i^{(b)k} >$, and geravitational, $< T_i^{(g)k} >$, parts are determined by (22) and (25) respectivly. In the regions $x < x_1$ and $x > x_2$ the boundary part is zero and only gravitational polarization part remains. The forces acting on plates are determined by boundary part only. The effective pressure created by gravitational part in (26) is equal to

$$p_{g1} = - < T_1^{(g)1} >= -\frac{\alpha^2 K^4(3\gamma^2 - 2\alpha^2)}{960\pi^2 a^4}A^{4(\alpha+\gamma)}(x) \quad (27)$$

and is the same from the both sides of the plates, and hence leads to the zero effective force. Vacuum boundary part pressures acting on plates are

$$p_{b1}^{(1,2)} = p_{b1}(x = x_{1,2}) = - < T_1^{(b)1}(x = x_{1,2}) >= -\frac{\pi^2 A^{4\alpha}(x_{1,2})}{480a^4} \quad (28)$$

and have attractive nature. The boundary part of the total energy between the plates can be found by standard way:

$$E_b = \int_{x_1}^{x_2}\int\int dxdydz\sqrt{-g} < T_0^{(b)0} >= -\frac{\pi^2}{1440a^3}\int\int dydz \quad (29)$$

where we have used the definition of a in accordance with (11). It can be easily seen that total energy (29) and pressures (28) are connected by standard thermodynamical relation

$$p_{b1}(x_1) = -\frac{dE}{dV_1}|_{x_2=const} = -\frac{dE}{dx_1dydz}A^{4\alpha+\gamma+1}(x_1)|_{x_2=const}, \quad dV_1 = A^{-4\alpha-\gamma-1}(x_1)dx_1dydz \quad (30)$$

and similar relation for $p_{b1}(x_2)$.

Only in the case of conformally invariant fields the eigenmodes have simple form (12) in the domain wall geravitational field (3). This is a direct consequnce of the conformally equivalence of the metric (3) to the Minkowskian one. Indeed by the coordinate transformation

$$x = f(x), \qquad (1 + K|f(x)|)^{-\gamma} = 1 - K\gamma|x| \tag{31}$$

It can be seen that the metric (3) takes a manifastly conformally flat form [7].

$$ds^2 = (1 - K\gamma|x|)^{\frac{2\alpha}{\gamma}}(dt^2 - dx^2 - dy^2 - dz^2) \tag{32}$$

Let $< T_{ik}^{(M)} >$ be the regularized standard energy-momentum tensor for a conformally coupled scalar field in the case of parallel plate counfiguration in flat space-time with metric ηik

$$< T_i^{(M)k} >= -\frac{\pi^2}{1440a^4} diag(1, -3, 1, 1) \tag{33}$$

where $a = x_2 - x_1$ and x_i are coordinates for the plates. Using the standard relation between the energy-momentum tensor for conformally coupled situation

$$< T_i^k[\tilde{g}_{lm}] >= (\frac{\eta}{\tilde{g}})^{\frac{1}{2}} < T_i^{k(M)}[\eta_{lm}] > -\frac{1}{2880\pi^2}\left[\frac{1}{6}^{(1)}\tilde{H}_i^k -^{(3)}\tilde{H}_i^k\right] \tag{34}$$

(the standard notations ${}^{(1,3)}H_i^k$ for some combinations of curvature tensor components see [8]), where tilde notes the quantities in the coordinate system $(t,\ X,\ y,\ z,)$ with metric (32). By making the transformation (see (31)) to the initial coordinate system $(t,\ x,\ y,\ z)$ from (34) we receive the result (26), where a is expressed via the coordinates x_1, x_2 of plates in system (3) by relation (11).

3. Concluding remarks

In this paper we calculate the casimir energy for conformally invariant scalar field in the standard parallel plate, on background of planar static domain wall. The boundary conditions over scalar field on the plate are Dirichlet boundary conditions. For calculating the vacuum expectation values of the energy-momentum tensor we use the mode sums method and Abel-Plana summation formula. The result contains two terms, one comes from the boundary conditions and the other one from the effect of gravitation over the vacuum of scalar filed. The quantity which reflects the effect of bondary conditions is finit. If we write this term in flat space-time limit we obtain the standard result of casimir effect for parallel plates. All divergences are in the part which describs the polarization of scalar vacuum by domain wall background in case of absence of boundaries. The effective pressure created by gravitational part is the same for both sides of the plates and hence leads to the zero effective force, but the vacuum boundary part pressures acting on plates is attractive. When the total energy between the plates are found, one can readily see that the total energy and pressures are connected through the standard thermodynamical relations. These results can easily be obtained using the conformal properties of the metric and the scalar field.

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